PEA
2019

1. Robinson Crusoe will live this period (period 1) and the next period (period 2) as the only inhabitant of his island completely isolated from the rest of the world. His only income is a crop of 100 coconuts that he harvests at the beginning of each period. Coconuts not consumed in the current period spoil at the rate of 20% per period. Crusoe's preference over consumption in period 1 (c_1) and consumption in period 2 (c_2) is given by the utility function $u(c_1, c_2) = \min\{5c_1, 6c_2\}$. Crusoe's utility maximizing consumption choice is given by

   (a) $c_1 = \frac{200\times6}{11}$, $c_2 = \frac{200\times5}{11}$.

   (b) $c_1 = 90$, $c_2 = 108$.

   (c) $c_1 = 100$, $c_2 = 100$.

   (d) none of the above.

2. The domestic supply and demand equations for a commodity in a country are as follows: Supply: $P = 50 + Q$, Demand: $P = 200 - 2Q$, where P is the price in rupees per kilogram and Q is the quantity in thousands of kilograms. The country is a small producer in the world market where the price (which will not be affected by anything done by this country) is Rs. 60 per kilogram. The government of this country introduces a "Permit Policy" which works as follows. The government issues a fixed number of Permits – each Permit allows its owner to sell exactly 100 kilograms of the commodity in this country's market. An exporter from a foreign country cannot sell this commodity in this country unless she purchases such a Permit. Suppose the government issues 300 Permits. What is the *maximum* price an exporter is willing to pay for a Permit?

   (a) Rs. 3000.

(b) Rs. 2000.

(c) Rs. 1500.

(d) Rs. 1000.

3. SeaTel provides cellular phone service in Delhi and has some monopoly power in the sense that it has its captive customer base with each customer's weekly demand being given by: $Q = 60 - P$, where Q denotes hours of cell phone calls per week and P is the price per hour. SeaTel's total cost of providing cell phone service is given by $C = 20Q$, so that the marginal cost is $MC = 20$. Suppose SeaTel offers a "*Call-As-Much-As-You-Wish*" deal: it charges only a flat *weekly access fee*, and once a customer pays the flat access fee, he/she can call as much as he/she wishes without paying any extra usage fee per hour. The *weekly access fee* that SeaTel should charge to maximize its profit is given by

   (a) 1800.

   (b) 1200.

   (c) 800.

   (d) 40.

4. A bus stop has to be located on the interval $[0, 1]$. There are three individuals located at points 0.2, 0.3 and 0.9 on the interval. If the bus stop is located at point x, then the utility of an individual located at y is $-|y - x|$, that is, the negative of the distance between the bus stop and the individual's location. A relocation of the bus stop is said to be *Pareto improving* if at least one individual is better off and no individual is worse off from the relocation. A location of the bus stop is said to be *Pareto efficient* if there does not exist any Pareto improving relocation. Then

(a) 0.5 is the only Pareto efficient location.

(b) $\frac{0.2+0.3+0.9}{3}$ is the only Pareto efficient location.

(c) Median of 0.2, 0.3 and 0.9 is the only Pareto efficient location.

(d) none of the above.

5. Consider three goods: (a) cable television, (b) a fish in international waters, and (c) a burger. Also consider four descriptions of the goods: (A) non-rival and non-excludable, (B) rival and excludable, (C) non-rival and excludable, and (D) rival and non-excludable. In what follows we match goods to possible descriptions. Choose the correct match.

(a) (a)-(A), (b)-(C), (c)-(B).

(b) (a)-(C), (b)-(D), (c)-(A).

(c) (a)-(C), (b)-(B), (c)-(A).

(d) (a)-(C), (b)-(D), (c)-(B).

6. Consider an economy consisting of three individuals – 1, 2 and 3, two goods – A and B, and a single monopoly firm that can produce both goods at zero cost. Each individual would like to buy exactly 1 unit of the goods A and B, if at all. An individual's *valuation* of a good is defined as the maximum amount she is willing to pay for one unit of the good. Individual 1's valuation of good A is Rs. 10 and that of good B is Rs. 1. Individual 2's valuation of good A is Rs. 1, and that of good B is Rs. 10. Individual 3's valuation is Rs. 7 for good A, and Rs. 7 good B. The firm can charge a single price p_A for good A, a single price p_B for good B, and a bundled price p_{AB} such that if an individual pays p_{AB} then she gets the bundle consisting of one unit each of

goods A and B. If the monopolist sets p_A, p_B and p_{AB} to maximize its profit then

(a) $p_A = 11$, $p_B = 11$, $p_{AB} = 11$.

(b) $p_A = 11$, $p_B = 11$, $p_{AB} = 14$.

(c) $p_A = 10$, $p_B = 10$, $p_{AB} = 11$.

(d) none of the above.

7. Consider a Bertrand duopoly with two firms, 1 and 2. Both firms produce the same good that has a market demand function $p = 10 - q$. The market is equally shared in case the firms charge the same price, otherwise the lower priced firm gets the entire demand. A firm must satisfy all the demand coming to it. The cost function of firm 1 is $3q_1$, that of firm 2 is $2q_2$. Suppose prices vary along the following grid, $\{0, 0.1, 0.2, \ldots\}$. The Bertrand equilibrium is given by

   (a) $p_1 = 2$, $p_2 = 2$.

   (b) $p_1 = 3$, $p_2 = 2$.

   (c) $p_1 = 3$, $p_2 = 2.9$.

   (d) $p_1 = 3$, $p_2 = 3$.

8. Consider a monopolist with a market demand function $p = 20 - q$. It is a multi-plant monopolist with two plants, plant 1 and plant 2, where the plant specific cost function of plant i , $i = 1, 2$, is

$$c_i(q_i) = \begin{cases} 2 + 4q_i, \text{ if } \ q_i > 0, \\ 0, \text{ otherwise.} \end{cases}$$

   The optimal monopoly profit is given by

   (a) 60.

(b) 64.

(c) 68.

(d) 62.

9. Consider a closed economy in which an individual's labour supply (L) to firms is determined by the amount which maximizes her utility function $u(C, L) = C^{\alpha}(1-L)^{\beta}$, where $\alpha, \beta > 0$, $\alpha+\beta < 1$, and C is consumption expenditure which is taken to be equal to wage income (wL). Then

(a) labour supply does not depend on the wage rate w.

(b) labour supply is directly proportional to the wage rate w.

(c) labour supply is inversely proportional to the wage rate w.

(d) more information is needed to derive the labour supply.

10. In the scenario described in Question 9, assume that the economy is Keynesian, that is, investment expenditure (I) is autonomous and output (Y) is determined by aggregate demand, $Y = C + I$. The aggregate production function is given by $Y = AL^{\theta}$, where $A > 0$ is a productivity parameter and $0 < \theta < 1$. [Note that the firm's employment of labour is obtained by equating the marginal product of labour to w.] Then the marginal propensity to consume is

(a) $\frac{\alpha+\beta}{\theta}$.

(b) $\frac{\beta}{\theta}$.

(c) α.

(d) θ.

11. Consider a Solow growth model (in continuous time) with a production function with labour augmenting technological change,

$Y_t = F(K_t, A_t L_t)$, where Y_t denotes output, K_t denotes the capital stock, A_t denotes the level of total factor productivity (TFP), and L_t denotes the stock of the labour force. Assume that L_t grows at the rate $n > 0$ and A_t grows at the rate $g > 0$, that is, $\frac{\dot{L}}{L} = n$ and $\frac{\dot{A}}{A} = g$, and the capital accumulation equation is given by $\dot{K} = sY_t - \delta K_t$, where $s \in [0, 1]$ is the exogenous savings rate, and $\delta \in [0, 1]$ is the depreciation rate of capital. [Note that for any variable x, $\dot{x}$ denotes $\frac{dx}{dt}$.] Define capital in efficiency units to be $Z \equiv \frac{K}{AL}$. Then the expression for $\frac{\dot{Z}}{Z}$ is given by

(a) $\frac{\dot{Z}}{Z} = \frac{sf(Z)}{Z} - (n + g)$.
(b) $\frac{\dot{Z}}{Z} = \frac{sf(Z)}{Z} - (\delta + n + g)$.
(c) $\frac{\dot{Z}}{Z} = \frac{sf(Z)}{Z}$.
(d) $\frac{\dot{Z}}{Z} = \frac{sf(Z)}{Z} - n$.

12. In the Solow growth model described in Question 11, the growth rate of Y at the steady state is given by

(a) $n + g$.

(b) $\delta + n + g$.

(c) zero.

(d) n.

13. Consider an IS-LM model where the IS curve is represented by $0.25Y = 500 + G - i$, and money demand function is given by $\frac{M}{P} = \frac{2Y}{e^i}$. The notations are standard: Y denotes output, G denotes government expenditure, i denotes the interest rate, P is the price level and e is the exponential. Suppose the government wants to increase spending and therefore the central bank decides to change the money supply accordingly such that the interest

rate remains the same in the short run. Then the change in money supply satisfies the following condition:

(a) $\frac{dM}{dG} = e$.

(b) $\frac{dM}{dG} = \frac{Ye}{M}$.

(c) $\frac{dM}{dG} = \frac{Y}{M}$.

(d) $\frac{dM}{dG} = \frac{4M}{Y}$.

14. An agent lives for two periods. Her utility from consumption in period 1 (c_1) and consumption in period 2 (c_2) is given by $u(c_1, c_2) = \log(c_1) + \beta \log(c_2)$, where $0 < \beta < 1$ is the discount factor reflecting her time preference. The agent earns incomes w_1 in period 1 and w_2 in period 2. The rate of interest is $r > 0$. The agent chooses c_1 and c_2 so as to maximize $u(c_1, c_2)$ subject to her budget constraint. Consider a *temporary* increase in income where w_1 increases but the agent does not change her expectations about w_2. Then the marginal propensity to consume of present consumption with respect to w_1, $\frac{dc_1}{dw_1}$, is given by

(a) $\frac{1}{1+\beta}\left(1 + \frac{1}{1+r}\right)$.

(b) $\left(\frac{1}{1+\beta}\right)\left(\frac{1}{1+r}\right)$.

(c) $\frac{1}{1+\beta}$.

(d) 1.

15. In the scenario described in Question 14, consider a *permanent* increase in income where w_1 increases and the agent expects that w_2 will also increase by the same amount. Then $\frac{dc_1}{dw_1}$ is given by

(a) $\frac{1}{1+\beta}\left(1 + \frac{1}{1+r}\right)$.

(b) $\left(\frac{1}{1+\beta}\right)\left(\frac{1}{1+r}\right)$.

(c) $\frac{1}{1+\beta}$.

(d) 1.

16. For what values of a are the vectors $(0, 1, a), (a, 1, 0), (1, a, 1)$ in $\mathbb{R}^3$ linearly dependent?

(a) 0.

(b) 1.

(c) 2.

(d) $\sqrt{2}$.

17. Which of the following set of vectors form a basis of $\mathbb{R}^2$?

(a) $\{(2, 1)\}$.

(b) $\{(1, 1), (2, 2)\}$.

(c) $\{(1, 1), (1, 2), (2, 1)\}$.

(d) $\{(1, 1), (2, 3)\}$.

18. If a candidate is good he is selected in MSQE examination with probability 0.9. If a candidate is bad he is selected in MSQE examination with probability 0.2. Suppose every candidate is equally likely to be good or bad. If you meet a candidate who is selected in the MSQE examination, what is the probability that he will be good?

(a) $\frac{11}{20}$.

(b) $\frac{9}{10}$.

(c) $\frac{9}{11}$.

(d) $\frac{11}{12}$.

19. Let $S_1 = \{2, 3, 4, \ldots, 9\}$. First, an integer s_1 is drawn uniformly at random from S_1. Then s_1 and all its factors are removed from S_1. Let the new set be S_2. Next an integer s_2 is drawn uniformly at random from S_2. Then s_2 and all its factors are removed from S_2. Let the new set be S_3. Finally, an integer s_3 is drawn uniformly at random from S_3. What is the probability that $s_1 = 2, s_2 = 3, s_3 = 5$?

(a) $\frac{1}{8}$.

(b) $\frac{1}{64}$.

(c) $\frac{1}{16}$.

(d) $\frac{1}{72}$.

20. Mr. A and B are independently tossing a coin. Their coins have a probability 0.25 of coming HEAD. After each of them tossed the coin twice, we see a total of 2 HEADS. What is the probability that Mr. A had exactly one HEAD?

(a) $\frac{2}{3}$.

(b) $\frac{1}{2}$.

(c) $\frac{1}{4}$.

(d) $\frac{1}{3}$.

21. Consider the following function $f : \mathbb{R} \to \mathbb{R}$.

$$f(x) = \begin{cases} x & \text{if } x \leq e \\ x(log_e x) & \text{if } x > e. \end{cases}$$

Which of the following is true for f?

(a) f is not continuous at e.

(b) f is not differentiable at e.

(c) f is neither continuous nor differentiable at e.

(d) f is continuous and differentiable at e.

22. Let $f : [-1, 1] \to \mathbb{R}$ be a continuous and weakly increasing function such that $\int_{-1}^{1} f(x)dx = 2\int_{-1}^{1} f(-x)dx$. Suppose $f(-1) = 0$, then $f(1)$ is

(a) 0.

(b) 1.

(c) $\frac{1}{2}$.

(d) none of the above.

23. Let $A \subseteq \mathbb{R}$ and $f : A \to \mathbb{R}$ be a twice continuously differentiable function. Let $x^* \in A$ be such that $\frac{\partial f}{\partial x}(x^*) = 0$. Consider the following two statements: (i) if $\frac{\partial^2 f}{\partial^2 x}(x^*) \leq 0$, then x^* is a point of local maximum of f; (ii) if x^* is a point of local maximum of f, then $\frac{\partial^2 f}{\partial^2 x}(x^*) < 0$. Which of the following is true?

(a) both (i) and (ii) are correct.

(b) both (i) and (ii) are incorrect.

(c) (i) is correct but (ii) is incorrect.

(d) (ii) is correct but (i) is incorrect.

24. Consider the function $f(x) = e^x$ for all $x \in \mathbb{R}$. Which of the following is true?

(a) f is quasi-convex.

(b) f is quasi-concave.

(c) f is neither quasi-convex nor quasi-concave.

(d) f is both quasi-convex and quasi-concave.

25. Consider the following matrix A.

$$A = \begin{bmatrix} x & 0 & k \\ 1 & x & k-3 \\ 0 & 1 & 1 \end{bmatrix}$$

Suppose determinant of A is zero for two distinct real values of x. What is the least positive integer value of k?

(a) 1.

(b) 9.

(c) 10.

(d) 8.

26. Define the following function on the set of all positive integers.

$$f(n) = \begin{cases} 2 \times 4 \times \ldots \times (n-3) \times (n-1) & \text{if } n \text{ is odd} \\ 1 \times 3 \times \ldots \times (n-3) \times (n-1) & \text{if } n \text{ is even.} \end{cases}$$

What is the value of $f(n+2)f(n+1)$?

(a) $n!$.

(b) $(n+1)!$.

(c) $(n+2)!$.

(d) $(n+2)(n!)$.

27. The sequence $\{x_n\}_{n \geq 0}$ is defined as follows. We set $x_0 = 1$ and $x_n = \sum_{j=0}^{n-1} x_j$ for each integer $n \geq 1$. Then the value of the expression $\sum_{j=0}^{\infty} \frac{1}{x_j}$ is equal to

(a) ∞.

(b) 2.

(c) 3.

(d) $\frac{7}{4}$.

28. For what values of p does the following quadratic equation have more than two solutions (variable in this equation is x)?

$$(p^2 - 16)x^2 - (p^2 - 4p)x + (p^2 - 5p + 4) = 0$$

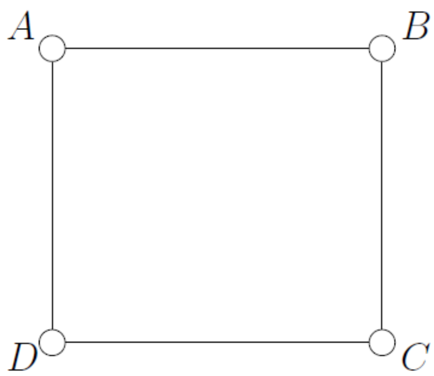
(a) No such value of p exists.

(b) -4 and 4.

(c) 1 and 4.

(d) 4.

29. Consider the square with vertices A, B, C, D as shown in the following figure. Call a pair of vertices in the square adjacent if they are connected by an edge in the figure. You have four colours: RED, BLUE, GREEN, YELLOW. How many ways can you colour the vertices A, B, C, D such that no adjacent vertices share the same colour?



(a) 84.

(b) 24.

(c) 72.

(d) 108.

30. Two players P_1 and P_2 are playing a game which involves filling the entries of an $n \times n$ matrix, where $n \geq 2$ is an **even** integer. Starting with P_1, each player takes turn to fill an unfilled entry of the matrix with a real number. The game ends when all entries are filled. Player P_1 wins if the determinant of the final matrix is non-zero. Else, player P_2 wins. A player $i \in \{1, 2\}$ has a **winning strategy** if irrespective of what the other player does, i wins by following this strategy. Which of the following is true?

(a) Player 1 has a winning strategy.

(b) Player 2 has a winning strategy.

(c) No player has a winning strategy.

(d) None of the above.

PEB

2019

Group A

1. [25 marks: $6+7+12$]

Ms. A earns Rs. 25,000 in period 1 and Rs. 15,000 in period 2. Mr. B earns Rs. 15,000 in period 1 and Rs. 30,000 in period 2. They can borrow money at an interest rate of 200%, and can lend money at a rate of 0%. They like both consumption in period 1 (C_1) and consumption in period 2 (C_2), and their preferences are such that their chosen consumption bundles will always lie on their budget lines.

(a) [6 marks]

Write down the equations of their budget constraints and draw their budget lines in the same figure by plotting consumption in period 1 (C_1) (in thousand rupees) on x-axis and consumption in period 2 (C_2) (in thousand rupees) on y-axis.

(b) [7 marks]

Given the income profile and the market interest rates, Mr. B chooses to borrow Rs. 5,000 in period 1.

Give an example of a consumption profile (that is, (C_1, C_2)) such that, if Ms. A chooses this profile, we would know for sure that Ms. A and Mr. B have **different preferences** for consumption in period 1 (C_1) and consumption in period 2 (C_2). Give a clear explanation for your answer.

(c) [12 marks: $6+6$]

Suppose now that Ms. A and Mr. B have the **same preferences** for C_1 and C_2, and, as in part (b), Mr. B borrows Rs. 5,000 in period 1.

i. Suppose that Ms. A chooses to be a lender in period 1. Find out, with a clear explanation, the *maximum* amount that she will lend in period 1 consistent with the fact that they have the same preferences for C_1 and C_2.

ii. Explain clearly whether Mr. B is better off than Ms. A.

2. [25 marks: $6 + 12 + 7$]

Consider an exchange economy with two agents 1 and 2 and two goods X and Y. There is one unit of both goods in the economy. An allocation is a pair $\{(X_1, Y_1), (X_2, Y_2)\}$ where $X_1 + X_2 = 1$, $Y_1 + Y_2 = 1$, and (X_1, Y_1) and (X_2, Y_2) are the consumption bundles of agents 1 and 2 respectively. The utility function for agent 1 is given by $u_1(X_1, Y_1) = X_1 \cdot Y_1$ and that of agent 2 by $u_2(X_2, Y_2) = 2X_2 + Y_2$.

(a) [6 marks]

Describe the set of Pareto-efficient allocations in the economy.

(b) [12 marks: $6 + 6$]

An allocation $\{(X_1, Y_1), (X_2, Y_2)\}$ is *envy-free* if no agent strictly prefers the consumption bundle of the other agent to her own, that is, $u_1(X_1, Y_1) \geq u_1(X_2, Y_2)$ and $u_2(X_2, Y_2) \geq u_2(X_1, Y_1)$.

i. Consider each of the two statements below. Decide whether they are true or false. Justify your answer with a proof or a counter-example as appropriate.

A. All Pareto-efficient allocations are envy-free.

B. All envy-free allocations are Pareto-efficient.

ii. Describe the set of envy-free allocations in the economy.

(c) [7 marks]

Suppose each agent has an endowment of half-unit of each good. Prove **without direct computation** that the competitive equilibrium allocation is both Pareto-efficient and envy-free.

3. [25 marks: $7+3+3+12$]

Two flat-mates, 1 and 2, rent a flat and play their own music on the only CD player owned by the flat-owner. They both like their own music, but dislike the music played by the other person. Given the timing constraints, each one must play her own music when the other person is also present. Let m_i denote the amount of music played by i, and Y_i denote her amount of money holding. Individual i's utility function is

$$u_i(m_1, m_2, Y_i) = 8m_i - 2m_i^2 - \frac{3}{2}m_j^2 + Y_i, \quad i, j = 1, 2, \ i \neq j.$$

(a) [7 marks]

How much music would each individual play? What is the efficient amount of music for each individual? Is the amount of music actually played more or less than the efficient level? Explain the economic intuition for your answer.

(b) [3 marks]

Suppose that individual 2 is considering to gift a headphone to her flat-mate on her birthday. Assume that she does not get any utility from just gift-giving. What is the maximum price she is willing to pay for the headphone?

(c) [3 marks]

Suppose that the price of the headphone is Rs. 11. Does it make sense for the two flat-mates to jointly buy a headphone, sharing the price equally, and making a binding commitment that they would each listen to their own music only via the headphone?

(d) [12 marks]

Now suppose that the CD player is owned by individual 1 so that she can prevent individual 2 from playing any music at all. Suppose individual 1 can offer a *take-it-or-leave-it* contract that looks like the following:

"I shall play music at a level $\bar{m}_1$, and you can play music at the level $\bar{m}_2$ in return for a sum of Rs. T."

In case the offered contract is rejected, individual 1 selects m_1 unilaterally, and individual 2 cannot play any music of her choice. Solve for the optimal levels of $\bar{m}_1$, $\bar{m}_2$ and T. Discuss the economic intuition for your answer.

Group B

1. [25 marks: $2 + 10 + 13$]

   Consider a country where there are only two provinces – A and B. The production function to produce a single output Y is given by $Y = F(N^A + N^B)$ where F is a concave function and N^i represents employees from province i, $i = A, B$. Wages paid to the employees are given by W^i, $i = A, B$. Price of the final good Y is denoted by P. The employers are price takers and take P, W^A and W^B as given.

   (a) [2 marks]

   Write down the expression for an employer's profit as a function of N^A and N^B, $\pi\left(N^A, N^B\right)$.

   (b) [10 marks]

   An employer chooses N^A and N^B to maximize

   $$u\left(N^A, N^B\right) = u\left(\pi\left(N^A, N^B\right), N^A, N^B\right),$$

   where $\frac{\partial u}{\partial \pi} > 0$, $\frac{\partial u}{\partial N^A} > 0$ and $\frac{\partial u}{\partial N^B} < 0$. The last two conditions on $u\left(N^A, N^B\right)$ imply that the employer prefers employees from province A but dislikes employees from province B.

   Write down the first order conditions for the employer's maximization problem assuming an interior solution.

   (c) [13 marks]

   In equilibrium do the employees from different provinces get the same wage? If yes, explain your answer. If not, then determine, with a clear explanation, which employees are paid more and by how much.

2. [25 marks: $4+5+8+8$]

Consider a concave utility function $u(c, l)$ where c represents consumption good and l represents labour supply (working hours, to be precise). While utility increases with the level of consumption good, increasing working hours reduces utility. Wage per hour of labour is given by w, thus working for l hours will ensure wl amount of total wage which is denoted by y, that is, $y = wl$. Given this, the utility function can be written as $u(c, \frac{y}{w})$. The price of the consumption good c is given by p. Also $\overline{L}$ is a fixed number of hours representing total time available to an agent and $\overline{L} - l$ represents leisure. [In all the figures you are asked to draw below, plot y on x-axis and c on y-axis.]

(a) [4 marks]

Derive the slope of an indifference curve for the utility function $u(c, \frac{y}{w})$ on the y-c plane.

(b) [5 marks]

Demonstrate the agent's utility maximizing choice of y and c in a figure by plotting her budget line and indifference curves for the utility function $u(c, \frac{y}{w})$.

(c) [8 marks]

Experiment 1: Suppose there is an increase in w. Demonstrate the agent's new utility maximizing choice of y and c in the same figure as in part (b). [Show clearly how the agent's budget line and/or indifference curves change as a result of the increase in w.] Compare the old and new choices with a brief economic explanation.

(d) [8 marks]

Experiment 2: Suppose, instead of an increase in w, there is a tax imposed on income. That is, the after-tax income of the agent is $(1 - \tau)\, y$ where τ is the proportional tax rate. In a new figure demonstrate the agent's new as well as old (as in part (b)) utility maximizing choices of y and c. [Show clearly how the agent's budget line and/or indifference curves change as a result of this proportional tax.] Compare the old and new choices with a brief economic explanation.

3. [25 marks: $4 + 3 + 10 + 4 + 4$]

Consider an agent who lives for three periods but consumes only in periods two and three where the consumptions are denoted by c_2 and c_3 respectively. Her utility is given by $u\,(c_2, c_3) = \log(c_2) + \beta \log(c_3)$, where $0 < \beta < 1$ is the discount factor reflecting her time preference. She invests an amount e in education in the first period which she borrows from the market at a given interest rate $r > 0$. Her income in the second period is $w \cdot h(e)$ where w is a fixed wage rate per unit of human capital and $h(e)$ is the amount of human capital that results from investment in education (e) in the first period. Assume that $h(e)$ is an increasing and concave function of e. The agent repays her education loan in the second period. She has no income in the third period. But she can save (s) in the second period from her income on which she receives the return $s\,(1 + r)$ in the third period to meet her consumption expenditure.

(a) [4 marks]

Write down the agent's period 2 and period 3 budget constraints separately.

(b) [3 marks]

Set up the agent's utility maximization problem by showing her choice variables clearly.

(c) [10 marks]

Write down the first order conditions for the agent's utility maximization problem.

(d) [4 marks]

Derive the ratio of consumptions in period 2 and period 3, $\frac{c_2}{c_3}$, in terms of the parameters of the model.

(e) [4 marks]

Explain how investment in education, e, depends on the preference parameter β.

Group C

1. [25 marks: $5 + 10 + 10$]

   Consider a street represented by the interval $[0, 1]$. Three agents, $\{1, 2, 3\}$, live on this street. Agent $i \in \{1, 2, 3\}$ lives at $x_i \in [0, 1]$, and assume that $x_1 \le x_2 \le x_3$. Suppose we locate a hospital at a point $p \in [0, 1]$.

   (a) [5 marks]

   We say p is **square-optimal** if it minimizes $\sum_{i=1}^{3}(x_i - p)^2$. Derive the square optimal value of p.

   (b) [10 marks]

   We say p is **absolute-optimal** if it minimizes $\sum_{i=1}^{3} |x_i - p|$.

   i. Argue that if p is absolute-optimal, then $p \in [x_1, x_3]$.

   ii. Use this to derive an absolute-optimal p.

   (c) [10 marks]

   Now suppose that n agents, $\{1, 2, 3, ..., n\}$, live on this street where $x_1 \le x_2 \le x_3 ... \le x_n$ and n is an odd number. Derive an absolute-optimal p.

2. [25 marks: $7 + 5 + 5 + 8$]

   Two random variables x_1 and x_2 are uniformly drawn from $[0, 1]$. Define the following function:

$$G(p) = p \times \text{Probability}[\ p \ge \max(x_1, x_2)] \qquad \forall\ p \in [0, 1].$$

   (a) [7 marks]

   Derive, with a clear explanation, the expression for $G(p)$.

(b) [5 marks]

Plot $G(p)$.

(c) [5 marks]

Is G convex or concave in p? Give clear explanations for your answer.

(d) [8 marks]

Find $\max_{p\in[0,1]} G(p)$.

3. [25 marks: $7+9+9$]

Let $X \subset \mathbb{R}$ and $f : X \to X$ be a continuous function.

(a) [7 marks]

Suppose $X = [0,1]$. By using the Intermediate Value Theorem, show that there exists $x^* \in X$ such that $f(x^*) = x^*$.

(b) [9 marks]

In each of the cases below, determine whether there exists $x^* \in X$ such that $f(x^*) = x^*$. Justify your claim by either providing a proof or a counter-example.

i. $X = (0,1)$ and f is continuous.

ii. $X = [0,1] \cup [2,3]$ and f is continuous.

iii. $X = [0,1]$ but f is not continuous.

(c) [9 marks]

Let $f_i : [0,1] \to [0,1]$, $i = 1,2,\ldots,m$, be a collection of m continuous functions. Prove that there exists $x^* \in [0,1]$ such that $\sum_{i=1}^{m} f_i(x^*) = mx^*$.